一、计算题（共 100 分）

1. Find the $PA = LU$ factorization for $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. ($L$: lower triangular matrix; $U$: upper triangular matrix) (10%)

2. For which numbers $c$ and $d$ does matrix $A$ have rank 2?

\[
A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}.
\] (10%)

3. Find the eigenvalues and the corresponding eigenvectors of

\[
A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}.
\] (10%)

4. Suppose a linear transformation $T$ transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$. Find $T(\boldsymbol{v})$ where $\boldsymbol{v} = (-1, 2)$. (10%)

5. Suppose $T$ projects every plane vector onto the 45° line. Find the standard matrix form for $T$. (10%)

6. Show that the set of all vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ satisfying the equations

\[
\begin{align*}
3x_1 - 2x_2 - x_3 - 4x_4 &= 0 \\
x_1 + x_2 - 2x_3 - 3x_4 &= 0
\end{align*}
\]

is a subspace of $\mathbb{R}^4$. Find a basis for this subspace. (10%)

7. Let $T(x_1, x_2, x_3) = (2x_1 + x_2 + 3x_3, 3x_1 - x_2 + x_3, -4x_1 + 3x_2 + x_3)$. Find a basis of $T(\mathbb{R}^3)$. Find a basis of Ker $T$. (10%)
8. Show that the quadratic form \( Q(x, y) = ax^2 + bxy + cy^2 \), \( a, b, c \in \mathbb{R} \) is positive definite if and only if \( a > 0 \) and \( b^2 - 4ac < 0 \). (15%)

9. Show that if \( \text{rank}(A) = n \) if and only if \( A \) is invertible, where \( A \) is an \( n \times n \) matrix. (15%)